Analysis III Final Examination, 2008, B.Math 2nd Year

Attempt all five questions. Each question carries 12 marks. You may consult books and notes. Results proved in class maybe cited without proof, but results of exercises must be derived in full.

1. (i): Let $U := GL(n, \mathbb{R})$ (an open subset of $M(n, \mathbb{R}) = \mathbb{R}^{n^2}$), and consider the map

$$\begin{array}{rcl} f : U & \to & M(n, \mathbb{R}) \\ A & \mapsto & A^{-1} \end{array} .$$

$-A^{-1}XA^{-1}$

Prove that $Df(A)X = \cancel{-A^{-2}X}$, for $X \in M(n, \mathbb{R})$. (*Hint:* Use the geometric series for $(I + tB)^{-1}$, for $B \in M(n, \mathbb{R})$ and t small).

(ii): Prove that there exists a C^∞ function $y = g(x)$, defined for x in some neighbourhood of 0 and satisfying (i) $g(0) = 1$ and (ii) $\cos xg(x) - g(x)^2 e^{-x^2} = 0$.

2. (i): Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map which is *line-preserving*, i.e. $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$ for all $x, y \in \mathbb{R}^n$, and all $t \in [0,1]$. Show that f is an affine map, viz. $f(x) = A.x + f(0)$ for some linear map $A : \mathbb{R}^n \to \mathbb{R}^m$. (*Hint:* Compute $Df(0)v$ for $v \in \mathbb{R}^n$).

(ii): Consider the function:

$$\begin{array}{rcll} f : \mathbb{R}^2 & \to & \mathbb{R} & \\ (x, y) & \mapsto & \dfrac{x^2 y}{x^2 + y^2} & \text{for } (x, y) \neq (0,0) \\ & \mapsto & 0 & \text{for } (x, y) = (0,0) \end{array} .$$

Show that both partial derivatives of f exist at $(0,0)$, however f is not differentiable at $(0,0)$.

3. (i): Consider the singular 2-cube in $\mathbb{R}^3 \setminus \{0\}$ given by:

$$\begin{array}{rcl} \sigma : [0,1] \times [0,1] & \to & \mathbb{R}^3 \setminus \{0\} \\ (s, t) & \mapsto & (\cos \pi (t - 1/2) \cos 2\pi s, \cos \pi (t - 1/2) \sin 2\pi s, \sin \pi (t - 1/2)) \end{array}$$

and the differential 2-form on $\mathbb{R}^3 \setminus \{0\}$ given by

$$\omega(x_1, x_2, x_3) = \frac{x_1 dx_2 \wedge dx_3 + x_2 dx_3 \wedge dx_1 + x_3 dx_1 \wedge dx_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}$$

(a) Show that ω is a closed form, and (b) compute $\int_\sigma \omega$.

(ii): On $\mathbb{R}^2$, consider the differential 2-form $\omega = x\, dx \wedge dy$, and the smooth vector fields $v = y^2 \partial_x + x^3 \partial_y$ and $w = 2x\partial_x - 3\partial_y$. Compute the smooth function $\omega(v, w)$.

4. (i): Let $U \subset \mathbb{R}^n$ be an open set, and define an operator

$$\begin{array}{rcl} \phi_i : \Omega^i(U) & \to & \Omega^{i+1}(U) \\ \omega & \mapsto & \omega \wedge dx_1 \end{array} .$$

Prove that $\omega \in \ker \phi_i$ iff $\omega \in \operatorname{Im} \phi_{i-1}$. (*Hint:* First prove that an i-form ω has a unique expression as $\omega = \alpha + \beta \wedge dx_1$ where $\alpha \in \Omega^i(U)$ and $\beta \in \Omega^{i-1}(U)$ do not involve dx_1).

(ii): On $U = \mathbb{R}^2 \setminus \{0\}$ consider the smooth functions:

$$P(x, y) = \frac{x - x^2 y - y^3}{(x^2 + y^2)^2}; \quad Q(x, y) = \frac{x^3 + y + xy^2}{(x^2 + y^2)^2} .$$

Show that there is no smooth function f on U satisfying the simultaneous first order PDE's:

$$\frac{\partial f}{\partial x} = P(x, y); \quad \frac{\partial f}{\partial y} = Q(x, y) .$$

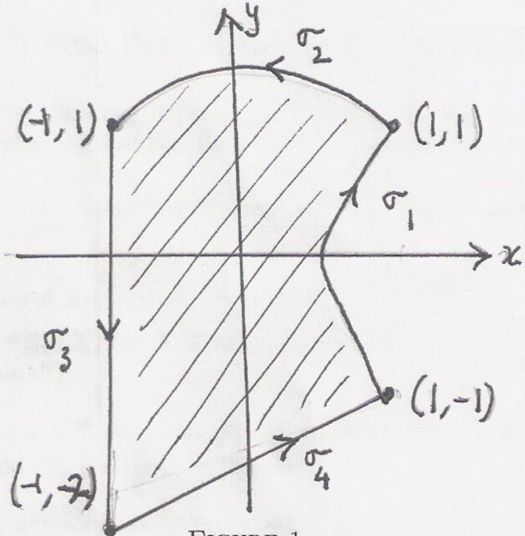

FIGURE 1

5. (i): Consider the 1-chain in $\mathbb{R}^2$ given by $\sigma = \sum_{i=1}^{4} \sigma_i$ where $\sigma_i : [0,1] \to \mathbb{R}^2$ are the singular 1-cubes given by:

$$\begin{aligned} \sigma_1(t) &= (2t^2 - 2t + 1, 2t - 1); \quad \sigma_2(t) = \left(\sqrt{2}\cos\frac{\pi(2t+1)}{4}, \sqrt{2}\sin\frac{\pi(2t+1)}{4}\right); \\ \sigma_3(t) &= (-1, 1 - 3t); \qquad \sigma_4(t) = (2t - 1, t - 2) \end{aligned}$$

(See Fig. 1). Compute the integral $\int_\sigma \omega$, where $\omega = xdy - ydx$.

(ii): Compute the area enclosed by σ (shaded region Fig. 1).